

# Financial Crises as Coordination Failures

Federico Ravenna\*

1 June 2014

## Abstract

We prove under what conditions the strategy of a large population of randomly coupled players engaging in a one-stage game converges in time to a Nash equilibrium, and if so, to which of the different ones. Extending Kandori, Mailath and Rob (1993) results, we introduce a flexible setup where the individual strategy choice is explicitly modeled, where the population dynamics is derived from the aggregation of the individual behaviour, and where a variety of selection and learning mechanisms can be easily introduced. This setup can be used for a novel explanation of financial and currency crises. The model generates herding among agents choice, with the same observational features of similar phenomena produced by social learning, information-driven models.

JEL Classification Numbers: F30; C72; C73

Keywords: Currency and Financial Crisis; Herd behaviour; Coordination game; Evolutionary game theory; Equilibrium selection

---

\*HEC Montreal, Department of Applied Economics. Email: federico.ravenna@hec.ca. I would like to thank Dan Friedman, Douglas Gale and Antonio Guarino for very helpful comments and suggestions.

# 1 Introduction

A rich literature in economics explains financial, banking and currency crises as the result of a coordination failure and of the existence of multiple equilibria (Obstfeld, 1986, Chang and Velasco, 1999). Coordination models establish the existence of multiple equilibria but do not provide a theory of equilibrium selection, nor can explain why or when a financial crisis will happen.

To address these shortcomings, several authors have modeled financial crises as the outcome of herding behaviour by investors who may rationally choose to ignore their own information and imitate the behaviour of other agents, generating an informational cascade (Calvo and Mendoza, 1996, Sachs, Tornell and Velasco, 1996, Chari and Kehoe, 1999). The herding result hinges on the following intuition. Agents decide sequentially and publicly what action to take - for example, whether to invest in a financial asset. If the first  $N$  players have received a negative private signal (imperfectly correlated with the real outcome) about the profitability of the investment, the  $(N+1)th$  player may choose to ignore his own information and follow the previous players' choice not to invest, based on the information revealed by previous actions. Since the action space is sufficiently coarse, complete information revelation is precluded, so the player may take a course of action different from the one she would take if information were publicly available (Banerjee, 1992, Bikhchandani and Sharma, 2001, Eyster and Rabin, 2010, Gale, 1996). Subsequent players are then more likely to herd, giving origin to an informational cascade. Morris and Shin (1995) discuss two important limitations of the informational herding theory of financial crises. First, it assumes that the payoff for the agent's choice is exogenous to the behaviour of the other agents, while often in financial crises asset prices depend on the actions of other agents. Second, informational herds happen because information is revealed only through the actions of the agents and private signals, while financial market participants have access to a large mass of publicly available information.

In this paper we build a sequential game based on the evolutionary game-theory setup of Kandori, Mailath and Rob (1993). We show that herding can occur as a result of coordination failure when agents play a best-reply strategy and agents have a small probability  $\varepsilon$  to choose a suboptimal strategy (a *mutation*). The model generates herding among agents' choices with the same observational features of herding produced by social learning, information-driven models. Agents coordinate the same strategy for a long time, and then suddenly and nearly contemporaneously they may change strategy and move to

the alternative equilibrium, for an extended period of time. The sudden swing between strategies happens with probability one and without any aggregate shock affecting the economy. As more and more agents herd on one choice, the probability of herding on the other choice is lower and lower.

Our approach builds on evolutionary game theory models where equilibrium selection across multiple Nash equilibria in 2x2 symmetric games is a limit outcome of a dynamic process (Banerjee and Weibull, 1992). We provide two original results.

First, as in Amir and Berninghaus (1996), we assume time is continuous and agents choices cannot happen simultaneously. This implies that the dynamics of a large population of repeatedly randomly coupled players engaging in a one-shot game can be described as a sequential game, where payoffs are determined in each interval  $dt$  when a single randomly selected agent re-optimizes its strategy. Compared to Amir and Berninghaus (1996), we introduce a flexible setup where the individual choice is explicitly modeled, where the population dynamics is derived from the aggregation of the individual behaviour, and where a variety of selection/learning mechanism can be easily introduced. We discuss the herding outcomes resulting from the long-run stable distribution for the best-reply strategy.

Second, we prove that as the probability of a mutation  $\varepsilon$  converges to zero, the population strategy converges to a unique history-independent equilibrium (the long run equilibrium). The long run equilibrium corresponds to the risk dominant one in the stage game. This result was first obtained by Kandori, Mailath and Rob (1993) using a replicator dynamics. While the literature generated by Kandori, Mailath and Rob (1993) has been moving towards dropping the replicator dynamics in favor of an explicit selection mechanism, the proposed models adopt a variety of setups making the results hard to compare. The setup we propose allows straightforward aggregation of individual behaviour, thus different hypotheses about the selection and choice mechanism are readily comparable.

The paper is structured as follows. Section 2 discusses related literature. Section 3 introduces the sequential game setup, build the birth-death Markov process describing the dynamics of strategy choice across the population, and provides an example of herding behaviour. Section 4 provides proofs of the equilibrium selection results as the mutation rate goes to zero. Section 5 concludes.

## 2 Related Literature

The framework we propose relies on a continuous-time evolutionary setup where the assumption that the option to change strategy *and* to play a game round arrives according to a random Poisson process implies the dynamics of agents' strategy has inertia, and mimics a sequential game, akin to the games in the informational herding literature. Spyrou (2014) provides a recent survey of herding in financial markets. Three observable features of informational herds have made this framework popular to model financial crises: 1) between crises there are extended periods of stability (agents do not continuously alter their choice); 2) changes in the choice of investors take the form of a market crash (once enough agents alter their choice, all will follow); 3) swings in investors' choices are protracted in time (if agents have all altered their initial choice, it will take a long time before they change it again).<sup>1</sup>

The theoretical setup of the sequential game is tightly related to the literature on equilibrium selection in evolutionary game theory. Most of this literature assume that players learn the equilibrium strategy in a naive way (the *bounded rationality* hypothesis). Two basic assumptions are commonly employed relative to the full rationality framework of maximizing and consistent behaviour. First, players suffer from a clearly spelled out kind of myopia: they act as if each stage of the game is the last rather than employing sophisticated repeated game strategies. Second, their strategy choice is affected by infinitesimally small random disturbances.

Kandori, Mailath and Rob (1993, KMR henceforth) and Foster and Young (1990) study the problem of whether the strategy of a large population of myopic randomly coupled players engaging in a one-shot game will converge in time to a Nash-equilibrium. KMR show that if random strategy mutations can occur, that is the strategy choice is affected by *unexpected* random noise, in 2x2 coordination games the population will always converge towards the risk-dominant equilibrium (Harsanyi and Selten, 1988).

KMR adopts the stochastic version of a Darwinian deterministic dynamic equation to describe the evolution of the number of players employing a given strategy. They show that the equilibrium selection result holds for any replicator dynamics satisfying certain weak assumptions. Bergin and Lipman (1996) show that the KMR results hinge on the

---

<sup>1</sup>Informational herding models can lead both to situations where all players are locked up in one equilibrium, or where the economy can jump between equilibria (Chamley, 1998: this phenomenon is known as the 'fragility' of informational cascades) - the result depends usually on whether signals have a finite or infinite support.

assumptions about the mutation’s stochastic process.

Subsequent work (Samuelson, 1993, Binmore et al., 1995, Amir and Berninghaus, 1996, Binmore and Samuelson, 1997, Sawa, 2011, Babichenko, 2012, Kreindelr and Young, 2013) has focused on adopting an explicit selection mechanism for the population strategy. Samuelson’s (1993) selection mechanism duplicates KMR’s setup. The explicit selection mechanism is a transition matrix whose probability entries are linked to payoffs. While achieving the goal of making explicit the stochastic process driving the population dynamics, the actual building of the matrix from individual behaviour involves a complicated aggregation procedure.

Bergin and Lipman (1996) and Binmore et al. (1995) examine the case of more complicated selection dynamics. Once deviating from the simple myopia assumption, hypothesis like noisy learning or payoff-dependent mutation rates make results dependent on the details of the model, and not directly comparable to KMR.

Amir and Berninghaus (1996) first introduced the continuous time framework to describe the selection mechanism (see also Binmore and Samuelson, 1997). While they characterize the evolution of the state of the population’s strategy choice as a birth-death Markov process, in the same way as in our paper, they build their transition probabilities as an aggregate process, rather than aggregating individual strategy choices.

### **3 A Sequential Coordination Game**

#### **3.1 Informal Discussion of Setup and Results**

Assume a finite number of agents engage in repeated plays of a coordination game. When agent  $i$  draws a random chance to optimize, she must choose between strategy  $s_1$  and  $s_2$ . The payoff is proportional to the average payoff obtained by playing a coordination game against all other agents, who have chosen their strategy in the past. Assume the agent chooses the average best response to the strategy of all the other players. At the end of the round, she collects the payoff and will get a chance to play another stage of the same game after a random time interval. The probability that she will have the chance to re-optimize arrives according to a Poisson process.

Without any added randomness, the game converges to an equilibrium where all agents play the same strategy. We are interested in how the outcome of the game changes if there is a small probability  $\varepsilon$  that each player makes a mistake when choosing the

strategy. Now a player's  $i$  strategy may deviate from the previous players' action.

The game in the long run features a probability distribution of equilibria. The existence of a stationary long run distribution can be proved. Moreover, for  $\varepsilon$  small enough it is typically bimodal, the two most likely states (denominated stable states) corresponding to the outcomes of the deterministic game where strategy-choice mistakes have zero probability to happen: either all players choose one strategy, or they all choose the alternative one. If some asymmetry is introduced in the payoffs, the stable state corresponding to the risk-dominant equilibrium of the deterministic game will be more likely to occur. In the coordination game where payoffs are asymmetric, if the probability of a mistake is driven to zero the risk-dominated equilibrium disappears: the long run equilibrium distribution in the limit assigns probability one to the risk-dominant equilibrium.

The outcome of the game resembles informational herds phenomena. Because players select actions sequentially, the system spends most of its time at or close to one of the stable states, and will eventually, and suddenly, swing to the other stable state. In fact, the system cannot move to the other stable state without going through all the states where agents' strategies differ. Since these states occur with negligible probability, the population will spend a negligible amount of time in these states, and swings between stable states will seldom happen. Yet, swings will happen with probability one as time goes by.

### 3.2 Stage Game Setup

A finite number of myopic players is matched at random time intervals to play a game. The state of the population is described by the vector:

$$\{n_1, n_2\}_t$$

where  $n_{i,t}$  is the number of players choosing strategy  $i$  at time  $t$ . The total number of agents is limited to  $N$ . The population state can be represented by a single variable  $n_1$  since:

$$n_2 = N - n_1$$

Players can be either matched tournament-style exactly once with each other player, or there can be an infinite number of random matches at each stage of the game. Time

flows continuously.

We make the same behavioral assumptions as KMR. In KMR agents choose myopic best replies (myopia hypothesis). Similarly, we assume agents do not take into account the far reaching implication of the behaviour - they play each stage of the game as if it were the last<sup>2</sup>. KMR also assumes that not all agents react instantaneously to the environment (inertia hypothesis). In our setup, each individual has to wait a random interval of time before being able to change their strategy.

When offered a chance to change strategy, players choose best responses given the information available. However, at each stage of the game all players in the KMR model may change strategy due to a random disturbance (mutation hypothesis). We assume that after an agent has decided on an action, there is a small probability that she alone will switch to some suboptimal alternative.

### 3.3 Individual Behaviour

Players face exactly the same stage game as in KMR, with generic payoff matrix:

	$s_1$	$s_2$
$s_1$	$a, a$	$b, c$
$s_2$	$c, b$	$d, d$

Given an initial condition, the agent must decide at each instant  $t$  what strategy to play, if she is given the chance to choose. Assume that the opportunity to choose follows a Poisson process with arrival rate  $k$ . This means that each agent  $l$  has a constant probability of getting the option to choose per infinitesimal time interval  $dt$  :

$$\begin{aligned}
 \Pr\{\text{switch options}_l(t, t + dt) = 0\} &= 1 - kdt + o(dt) \\
 \Pr\{\text{switch options}_l(t, t + dt) = 1\} &= kdt + o(dt) \\
 \Pr\{\text{switch options}_l(t, t + dt) > 1\} &= o(dt)
 \end{aligned} \tag{1}$$

and that the event  $\{\text{number of switch options}_l(t, t + dt)\}$  is completely independent of occurrences in  $(t_0, t)$ . Assume as well players are matched to play a stage game whenever any agent  $l$  gets the option switch.

---

<sup>2</sup>The behaviour is naive in two ways: players do not understand that their own behaviour may affect future plays of their opponents, and do not take into account the fact that their opponents are similarly adjusting their behaviour (Mailath, 1998).

The average expected payoff for a player who will choose strategy 1 is:

$$U_1[n_1^e(n_1)] = U_1(n_1) = \frac{a(n_1^e - 1)}{N - 1} + \frac{b(N - n_1^e)}{N - 1}$$

while choosing strategy 2 will yield:

$$U_2[n_2^e(n_1^e)] = U_2(n_1) = \frac{cn_1^e}{N - 1} + \frac{d(N - n_1^e - 1)}{N - 1}$$

where  $n_i^e$  is the total number of players expected to play strategy  $i$  at the current stage of the game.

A player is always already employing a strategy. The choice of which strategy to play next can thus be equated to the choice whether to change or not current strategy.

If in  $t$  the agent is offered the opportunity to switch, she knows there will be no other switch in  $dt$ , since  $\Pr\{\text{switch options}(t, t + dt) > 1\} = o(dt)$ . Deciding the optimal allocation is then a simple matter of comparing the payoffs gained by each strategy at time  $t$ . Because of the Poisson assumption, players optimize sequentially, the only information needed to calculate  $n_i^e$  is the own choice of the player. Assume WLOG the player is currently employing strategy 2. It will be optimal to switch if:

$$U_1(n_1) > U_2(n_1)$$

### 3.3.1 Mutations: baseline paradigm and variations

With probability  $\epsilon$ , the player will flip the optimal (myopic best reply) strategy decision just taken<sup>3</sup>. Therefore the probability of switching strategy if playing strategy  $j$  (and given the chance to switch) can be computed as:

$$\begin{aligned} \Pr\{\mu &= 1\} = \text{sign}[U_i(n_1) - U_j(n_1)](1 - \epsilon) + \epsilon[1 - \text{sign}[U_i(n_1) - U_j(n_1)]] \quad (2) \\ \text{sign}[x] &= 1 \quad \text{for } x \geq 0 \\ \text{sign}[x] &= 0 \quad \text{for } x < 0 \end{aligned}$$

---

<sup>3</sup>In KMR(1993) the player can die with probability  $\epsilon$  and the player replacing him will choose either strategy with equal probability. We assume implicitly that the new player will make the sub-optimal choice with probability 1. This assumption does not change any of the results while streamlining the calculations.

While in the rest of the paper we will assume equation 2 holds, a variety of alternative individual choice models could be selected.

**A. Mutations may happen before a best reply is selected.** With probability  $\epsilon$  player will not choose a best reply but simply flip strategy. Therefore the probability that strategy is switched if playing strategy  $j$  (and given the chance to switch) is:

$$\Pr\{\mu = 1\} = \epsilon + \text{sign}[U_i(n_1) - U_j(n_1)](1 - \epsilon)$$

The real process the equation describes might be as follows. As the chance to change strategy occurs to a player, she will die and be replaced with probability  $\epsilon$ . In this case, the new player will change strategy anyhow. With probability  $(1-\epsilon)$ , the player survives and gives best response. Note that if strategy  $i$  is a best reply, it will be chosen with probability 1.

**B. Noisy selection process.** Assume mutations occur before a best reply is selected, as in the previous model. Assume as well that if a player does not mutate, there is still only a probability  $p$  that she will adopt the best response. Then:

$$\Pr\{\mu = 1\} = \epsilon + \text{sign}[U_i(n_1) - U_j(n_1)] p (1 - \epsilon)$$

This individual choice mechanism replicates Binmore et al. (1995). Even as the mutation rate  $\epsilon$  is driven to zero, still the best reply will be adopted only a fraction  $p$  of the times when it is optimal. The introduction of a noisy selection mechanism has two consequences. First, it is possible to reduce the total degree of inertia of the system by lowering the value of  $k$  while at the same time keeping inertia in the best reply mechanism. Second, expected waiting times to move from an equilibrium to the other may be drastically reduced. In KMR, the only hope to escape the basin of attraction of an equilibrium is that a series long enough of mutations takes place to reach the basin of attraction of the other equilibrium. With a noisy selection process, one mutation only is required to move the system away from one of the long run equilibria. Once this mutation has occurred, the noisy learning process can cause more players to switch. This event may be more likely to happen than a sequence of mutations as the mutation rate is driven to zero.

**C. Aspiration and imitation model.** Instead of comparing the two payoffs and choose the strategy yielding the higher one, a player compares the payoff of the current strategy choice with an aspiration level, and changes strategy if it falls short. Having abandoned the strategy, she will imitate the strategy of the randomly selected player  $l$ , unless she mutates and chooses the opposite strategy. No mutation can occur if the player does not find it worthy to review its strategy upon comparing the current payoff with the aspiration level. Binmore and Samuelson (1997) propose such a selection mechanism, although their aggregation procedure is flawed (see the Appendix). Calling  $\Delta$  the aspiration level, the probability that a player switches strategy if playing strategy  $j$  (and given the chance to switch) is:

$$\Pr\{\mu = 1\} = \text{sign}[\Delta - U_j(n_1)] \left[ (1 - \varepsilon) \frac{n_i}{N-1} + \varepsilon \frac{(N - n_i - 1)}{N-1} \right]$$

The selection mechanism implies that the agent who finds it worthy to switch strategy will do so if: 1) she chooses to imitate an agent playing strategy  $i$  (and the probability that a randomly picked agent will be playing strategy  $i$  is  $\frac{n_i}{N-1}$ ) and she does not mutate ( $\varepsilon$  being the mutation probability), or 2) she chooses to imitate an agent playing strategy  $j$  (and the probability that a randomly picked agent will be playing strategy  $j$  is  $\frac{N-n_i-1}{N-1}$ ) but she mutates.

**D. Payoff-dependent probability of best-reply adoption.** A player will choose always a best reply strategy, if she does not mutate. The probability of such a choice is directly proportional to the payoff difference. The probability that a player switches strategy if playing strategy  $j$  (and given the chance to switch) is:

$$\Pr\{\mu = 1\} = \varepsilon + \text{sign}[U_i(n_1) - U_j(n_1)] \left[ \frac{U_i(n_1) - U_j(n_1)}{\max_{n_1} \{U_i(n_1) - U_j(n_1)\}} \right] (1 - \varepsilon)$$

This individual choice mechanism replicates Amir and Berninghaus (1996). The intuition is that while a player may sometimes discard a best reply, she is less likely to commit such a mistake the higher the (positive) payoff difference is. Note that the process is asymmetric: if a strategy is not a best reply, the chance that it will be adopted is always zero (except if a mutation occurs).

**E. Payoff-dependent mutation rates.** Assume mutations represent random exper-

imentation carried out by the player. It is then difficult to reconcile with the assumption that the mutation rate is constant across states, across agents and over time. If the mutation rate is a function  $\epsilon = \epsilon(\frac{n_1}{n_2}) = \epsilon(n_1)$  of the current choice distribution across the population, the probability that strategy is switched if playing strategy  $j$  (given the chance to switch) is:

$$\Pr\{\mu = 1\} = \text{sign}[U_i(n_1) - U_j(n_1)][1 - \epsilon(n_1)] + \epsilon(n_1)[1 - \text{sign}[U_i(n_1) - U_j(n_1)]]$$

This observation is advanced in Bergin and Lipman (1996), where is proven that the KMR result holds only if the mutation rates for every state of the population converge to zero at the same rate.

### 3.4 Aggregate Behaviour of the Population

At each point in time the probability that in the next infinitesimal interval  $(t, t + dt)$  one of the  $n_{i,t}$  agents is offered an opportunity to switch strategy can be computed from equation 1 :

$$\begin{aligned} \Pr\{\text{switch options } (t, t + dt) = 1\} &= \sum_{l=1}^{n_{i,t}} \Pr\{\text{switch options}_l (t, t + dt) = 1\} \\ &= n_{i,t}(kdt + o(dt)) \end{aligned} \quad (3)$$

Then, the probability that a switch from strategy  $i$  to  $j$  will happen in the interval  $(t, t + dt)$  is, combining (3) and (2):

$$\begin{aligned} \Pr\{\text{switch from strategy } i \text{ to } j (t, t + dt) = 1\} &= \\ \Pr\{\text{switch options } (t, t + dt) = 1\} * \Pr\{\mu = 1\} &= \end{aligned}$$

$$n_{i,t}kdt * \{\text{sign}[U_j(n_1) - U_i(n_1)](1 - \epsilon) + \epsilon[1 - \text{sign}(U_j(n_1) - U_i(n_1))]\} + o(dt) \quad (4)$$

Players get a chance to change strategy according to a Poisson process with parameter  $k$  : as this parameter goes to infinity inertia goes to zero.

The evolution of the aggregate economy across time can now be described as a generalized birth-death Markov process in discrete state space  $\{n_1, n_2\}$  in continuous time. Equation 4 provides the probability rules necessary to characterize the specific jump Markov process (Gillespie, 1992):

$$\Pr\{n_{i,t+dt} = q \mid n_{i,t} = g\} =$$

$n_{j,t}kdt\{sign[U_i(n_1) - U_j(n_1)](1 - \epsilon) + \epsilon[1 - sign(U_i(n_1) - U_j(n_1))]\} + o(dt)$	if $q = g + 1$
$n_{i,t}kdt\{sign[U_j(n_1) - U_i(n_1)](1 - \epsilon) + \epsilon[1 - sign(U_j(n_1) - U_i(n_1))]\} + o(dt)$	if $q = g - 1$
$1 - n_{i,t}kdt\{sign[U_j(n_1) - U_i(n_1)](1 - \epsilon) + \epsilon[1 - sign(U_j(n_1) - U_i(n_1))]\} - n_{j,t}kdt\{sign[U_i(n_1) - U_j(n_1)](1 - \epsilon) + \epsilon[1 - sign(U_i(n_1) - U_j(n_1))]\} + o(dt)$	if $q = g$ otherwise

From the above equations we define the probability transition rates (per infinitesimal interval  $dt$ ) of a jump in the state of the population from  $\{n_i, n_j\}$  to  $\{n_i + 1, n_j - 1\}$  occurring in the interval  $(t, t + dt)$ , equivalent to the probability transition rate of a player switching from strategy  $j$  to  $i$ , as:

$$w_{ij} = n_{j,t}k\{sign[U_i(n_1) - U_j(n_1)](1 - \epsilon) + \epsilon[1 - sign(U_i(n_1) - U_j(n_1))]\} \quad (5)$$

The transition rates are function of the state of the economy<sup>4</sup>.

---

<sup>4</sup>It is straightforward to define transition rates as function of the current state  $n_{1,t}$  rather than of the future state  $n_{1,t+1}$ . As can be readily seen,

$$\begin{aligned} w_{12} &= w_+(n_1) = (N - n_1)k\{sign[U_1 - U_2](1 - \epsilon) + \epsilon[1 - sign(U_1 - U_2)]\} \\ w_{21} &= w_-(n_1) = n_1k\{sign[U_2 - U_1](1 - \epsilon) + \epsilon[1 - sign(U_2 - U_1)]\} \end{aligned}$$

and

$$U_1 - U_2 = U_+ = \left[\frac{an_1}{N-1} + \frac{b(N-n_1-1)}{N-1}\right] - \left[\frac{cn_1}{N-1} + \frac{d(N-n_1-1)}{N-1}\right] \quad (6)$$

$$U_2 - U_1 = U_- = \left[\frac{c(n_1-1)}{N-1} + \frac{d(N-n_1)}{N-1}\right] - \left[\frac{a(n_1-1)}{N-1} + \frac{b(N-n_1)}{N-1}\right] \quad (7)$$

### 3.5 Law of Motion for the Population Probability Distribution

The Chapman-Kolmogorov backward equation gives the law of motion for the probability distribution  $\Pr(n_1, t)$  over the population states, that is the probability that at time  $t$  a fraction  $\frac{n_1}{N}$  of the population is adopting strategy  $s_1$  (Weidlich, 1991):

$$\frac{d\Pr(n_1; t|n_0; t_0)}{dt} = [w_{21}(n+1)\Pr(n+1; t) + w_{12}(n-1)\Pr(n-1; t)] - [w_{12}(n)\Pr(n; t) + w_{21}(n)\Pr(n; t)] . \quad (8)$$

where  $n = n_1$ . The interpretation of the master equation (8) is straightforward. The probability of state  $\{n\}$  can change with time because players may change their choice. The terms with positive signs are probability inflows from the neighbouring states  $\{n+1\}$  and  $\{n-1\}$ , while the terms with negative signs represent probability outflows from state  $\{n\}$  into neighbouring states.

### 3.6 Herding

We provide a simple example for a payoff matrix:

	$s_1$	$s_2$
$s_1$	$a, a$	$0, 0$
$s_2$	$0, 0$	$d, d$

with  $a, d > 0$ . Eqs. 9 and 10 can be used to build the stationary probability distribution of the sequential game for non-zero values of  $\varepsilon$ . When the mutation rate is large enough, most of the time half of the population will be playing one strategy and the other half will be playing the other strategy (figure 1).

Herding occurs for smaller, non-zero values of the mutation probability. Figure 2 shows the shape of the stationary probability distributions as  $\varepsilon$  becomes small and  $a = d$ . Agents will herd on one choice for an extended period of time, but there is always a positive probability that they will suddenly switch to the other strategy. When that happens, they will herd on the alternative choice. The probability of states where the strategy choice is equidistributed across the population is very small. If  $a \neq d$ , herding will mostly occur on one of the two equilibria, the Pareto-dominant one (figure 3) since the two alternative strategies have identical security levels.

Finally, we can obtain even sharper herding results by making the mutation probability state-dependent:  $\varepsilon_i = \varepsilon_i(n_i)$ . If  $\varepsilon_i$  is a decreasing function of the number of agents who are currently playing strategy  $i$ , a switch to strategy  $j$  is less and less likely as the number of agents herding on one choice increases.

Since the stationary distribution in eq. 9 depends on the ratio of the probability transition rates, the parameter  $k$  is irrelevant for the stationary distribution. We will show in the following that the arrival rate - the speed at which players receive a chance to choose a strategy - is irrelevant also for the limiting distribution results, as the mutation rate approaches zero. What is key for the results is not how much inertia the system has, but rather the assumption that, as close in time as they can choose, two players can never change strategy simultaneously<sup>5</sup>. On the other hand, the expected waiting time for the system to move from one state to another is affected by the parameter  $k$ .

## 4 Long Run Behaviour

For some temporally homogeneous birth-death Markov processes, the Markov state density function which solves equation 8 approaches a well behaved function as  $t \rightarrow \infty$ . Such processes are said to be stable, and the limit function  $\text{Pr}_s(n_1)$  is called the stationary Markov state density function. The birth-death stability theorem (Gillespie, 1992) provides conditions for the stability of a birth-death Markov process. Assume WLOG that  $n = n_1$ .

**Theorem 1** *Existence and uniqueness of the stable distribution. The Markov process defined by 5 is stable and satisfies:  $\lim_{(t-t_0) \rightarrow \infty} \text{Pr}(n; t | n_0, t_0) = \text{Pr}_s(n)$  for all  $n_0 \in [0, N]$*

**Proof.** See the appendix. ■

The birth-death stability theorem provides the exact functional form of the stable distribution  $\text{Pr}_s(n)$  (Gillespie, 1992, Weidlich, 1991):

---

<sup>5</sup>This is one of the reasons why Burdzy, Frankel and Pauzner (1998) obtain different limiting result. In their setup, more than one player can change strategy simultaneously.

$$\begin{aligned}
P_s(n) &= K \text{ if } n = 0 \\
P_s(n) &= K \prod_{j=1}^n \frac{w_{12}(j-1)}{w_{21}(j)} \text{ if } n = 1, 2, \dots, N
\end{aligned} \tag{9}$$

$$K = \left(1 + \sum_{n=1}^N \prod_{j=1}^n \frac{w_{12}(j-1)}{w_{21}(j)}\right)^{-1} \tag{10}$$

## 4.1 Equilibrium selection

The previous theorem can be usefully exploited to examine the long run equilibrium of the game as the probability of mutation converges to zero.

**Definition 2** *The limiting distribution.* When it exists, the probability measure  $P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n)$  where the pointwise limit is taken at the integer values  $n \in [0, N]$ , will be called the limiting distribution.

### 4.1.1 Symmetric 2x2 games

We will characterize the limiting distribution, and solve the equilibrium selection problem when it arises, for all games in the 2x2 symmetric class, except games which have only a mixed-strategy Nash equilibrium.

Only three Nash equilibria combinations can arise in 2x2 symmetric games, given the payoff matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

**Case 1:** if one strategy is strictly dominant, there will be only one Nash equilibrium in the game. This occurs when  $a > c$  and  $b > d$ , therefore strategy 1 strictly dominates strategy 2, or when  $a < c$  and  $b < d$ , implying strategy 2 strictly dominates strategy 1.

**Case 2:** if the game is a coordination one,  $a > c$  and  $b < d$ . Then there will be two strict Nash equilibria  $(a, a)$  and  $(d, d)$ , and one mixed strategy equilibrium.

**Case 3:** if  $a < c$  and  $b > d$  the game has only a symmetric Nash equilibrium in mixed strategies  $p^* = (d - b) / [(d - b) + (a - c)]$ .

The next sections provide the long run equilibrium convergence result in each of these cases. It is useful to establish the relationships between strictly dominant strategies, risk dominant strategies and the average payoffs  $U_i$ .

**Dominant strategies and average payoffs** First, note that  $U_i(n_1)$  is a linear function, therefore  $U_1$  and  $U_2$  can cross at most once (unless  $U_1 = U_2 \forall n_1$ ). Assume  $s_1$  is the dominant strategy. Then equation 6 implies that:

$$U_1(0) - U_2(0) = \frac{b(N-1)}{N-1} - \frac{d(N-1)}{N-1} > 0$$

For  $N$  large enough, or for  $(a-c)$  enough bigger than  $(b-d)$ , equation 6 implies that:

$$U_1(N) - U_2(N) = \frac{aN-b}{N-1} - \frac{cN-d}{N-1} = \frac{N(a-c) - (b-d)}{N-1} \geq 0$$

If the previous inequality holds, then:

$$s_1 \text{ is dominant strategy} \leftrightarrow U_1(n_1) \geq U_2(n_1) \quad \forall n_1 \quad (11)$$

while the inequality is reversed if the dominant strategy is  $s_2$ .

**Risk Dominance and average payoffs** In 2x2 coordination games, the equilibrium  $(a, a)$  risk dominates  $(d, d)$  if  $(a-c) > (d-b)$ . If the inequality is reversed, the equilibrium  $(a, a)$  will be the risk dominated one. Risk dominance may conflict with payoff dominance.

A useful relationship can be established between risk dominant strategies and average payoffs too. Let  $z^*$  be the value of the state variable  $n_1$  for which the average payoff of strategy 1 equals the average payoff of strategy 2. Using equation 6 this critical value is given by:

$$\begin{aligned} U_1 - U_2 &= U_+ > 0 \\ n_1 &> (N-1) \frac{(d-b)}{(a-c) + (d-b)} = z^* \end{aligned} \quad (12)$$

It is easy to see that

$$s_1 \text{ is risk dominant strategy} \leftrightarrow z^* < \frac{N-1}{2} \quad (13)$$

while the opposite inequality holds if the risk dominant strategy is  $s_2$ . Intuitively, these conditions ensure that the average payoff of a strategy which is risk dominant is higher than the average payoff of the alternative one for a larger portion of the state space.

Note that regardless of which is the risk dominant strategy, in coordination games equation 6 implies that

$$\text{Case 2} \leftrightarrow U_1(0) < U_2(0) \tag{14}$$

The sign of the inequality 12 is reversed in games having only a mixed strategy equilibrium, implying that

$$\text{Case 3} \leftrightarrow U_1(0) > U_2(0)$$

**The Average Profitability Criterion** Calculating the critical value  $z^*$  from equation 7 yields a definition slightly different from equation 12:

$$\begin{aligned} U_2 - U_1 &= U_- < 0 \\ n_1 &> N \frac{(d-b)}{(a-c) + (d-b)} - \frac{(a-c)}{(a-c) + (d-b)} = z^{*'} \end{aligned}$$

For  $z^{*'}$  to be approximately equal to  $z^*$  we need  $N$  large enough, or  $(a-c)$  enough bigger than  $(d-b)$ . This is the same regularity condition needed to ensure that a payoff dominant strategy yields higher average payoff for any value of the state variable  $n_1$ . This regularity condition is met when strategies' payoff satisfy the Average Profitability Criterion (Amir and Berninghaus, 1996):

**Definition 3 APC.** *The family of functions  $U$  satisfies the Average Profitability Criterion if:*

- 1)  $U_i(n_1) > U_j(n_1)$  for all  $n_1 \in [0, N]$  if and only if strategy  $s_i$  strictly dominates strategy  $s_j$ ;
- 2)  $U_i(n_1) = U_j(n_1)$  for all  $n_1 \in [0, N]$  if and only if strategy  $s_i$  is identical to strategy  $s_j$ .

We do not strictly need the function  $U_i$  to satisfy APC to prove the convergence theorems. The APC assumption makes the intuition behind the results very straightforward, so we will assume it holds in the following sections.

## 4.2 Games with a Dominant Strategy

If the stage game has a unique Nash equilibrium, we expect a population made up of (myopic) optimizing agents to converge to the equilibrium as their choice gets less and less noisy. This conjecture proves true as the following theorem illustrates.

**Theorem 4** *Suppose the stage game has a dominant strategy. For any population size  $N \geq 2$  the limiting distribution  $P_s^*(n)$  exists and is unique. It puts probability one on  $N$  if  $s_1$  is the dominant strategy and on 0 if  $s_2$  is the dominant strategy. That is, in the long run the whole population will play the dominant strategy as the noise affecting the individual choice gets smaller and smaller.*

**Proof.** See the appendix. ■

## 4.3 Coordination Games

Suppose that in a coordination game  $a > d$ , that is, the game is one of common interest. If players of the one stage game could coordinate, they would certainly play  $(a, a)$ , which is the Pareto-dominant equilibrium. If coordination is not possible, it is not so obvious that the Pareto-dominant equilibrium will be played, as it may call for a riskier strategy, in the sense defined by Harsanyi and Selten (1998). The next theorem proves that the Pareto dominant equilibrium is the long run equilibrium of coordination games only if the two strategies have identical security levels, that is  $b = c$ . If the payoff dominant equilibrium involves a higher risk, the risk dominant equilibrium will be selected instead.

**Theorem 5** *Suppose the stage game is a coordination game. For any population size  $N \geq 2$  and any  $z^* \neq \frac{N-1}{2}$  the limiting distribution  $P_s^*(n)$  exists and is unique. It puts probability one on  $N$  if  $s_1$  is the risk dominant strategy and on 0 if  $s_2$  is the risk dominant strategy. That is, in the long run the whole population will play the risk dominant strategy as the noise affecting the individual choice gets smaller and smaller.*

**Proof.** See the appendix. ■

If the two equilibria have identical security level, the Pareto dominant equilibrium will be as well the risk dominant one. We can then state the following corollary.

**Corollary 6** *Suppose the stage game is a coordination game and its two equilibria have identical security levels ( $b = c$ ). For any population size  $N \geq 2$  the limiting distribution*

$P_s^*(n)$  exists and is unique. It puts probability one on  $N$  if  $(a, a)$  is the payoff dominant equilibrium and on 0 if  $(d, d)$  is the payoff dominant equilibrium.

#### 4.4 Games of pure coordination

When  $z^* = \frac{N-1}{2}$ , none of the two Nash equilibria risk-dominates the other. We might expect that in this case the population will converge to the Pareto-dominant equilibrium. This is not the case. Instead, the following theorem applies. The flavor of the result is clearer when we note that the case  $z^* = \frac{N-1}{2}$  includes games of pure coordination. In these games,  $b = c$  and  $a = d > b = c$ . These conditions on the payoff matrix imply that 1) the risk-dominant equilibrium is as well Pareto-dominant; 2) none of the two Nash equilibria risk-dominates the other; 3) the two Nash equilibria  $E_1 = (s_1, s_1)$  and  $E_2 = (s_2, s_2)$  are both Pareto-efficient. An example of a pure coordination game is:

	$s_1$	$s_2$
$s_1$	1, 1	0, 0
$s_2$	0, 0	1, 1

Whether any kind of coordination is possible or not, there is no reason why the players would prefer one of the two completely symmetric Nash equilibrium outcomes. In fact, we will show that both equilibria are long run equilibria, in the sense that after a long enough period of time there is an equal chance that either one of the equilibria will be observed as  $\varepsilon \rightarrow 0$ . Perhaps more surprisingly, the result holds even if one of the two equilibria is Pareto-dominant, provided none is risk dominant.

**Theorem 7** *Suppose the stage game is a coordination game and  $z^* = \frac{N-1}{2}$ . For any odd population size  $N \geq 2$  the limiting distribution  $P_s^*(n)$  exists and is unique. It puts probability  $1/2$  on  $N$  and  $1/2$  on 0.*

**Proof.** See the appendix. ■

The meaning of the theorem deserves some thought. Contrary to the previous cases, the long run equilibrium distribution is bimodal. In the long run the whole population will play either of the two strategies as the noise affecting the individual choice gets smaller and smaller, yet will never lock on one. While individual players will keep switching strategies forever, it will become less and less likely that two strategies are played *at the same time* within the population. In the limit, we have a zero probability of observing

the system in a state where part of the population is playing one strategy, and part a different one. At the same time, we have to assign equal unconditional probability to the event that the population will all be playing one of the two Nash equilibrium strategy. The paradox is that because the stochastic process we are considering can only move one step at a time, there *must* be times when not all the population is playing the same strategy, while the system moves from one mode to the other.

## 5 Conclusions

This paper showed that herding can occur as a result of coordination failure in an evolutionary game theory setup analogous to the one proposed by Kandori, Mailath and Rob (1993). The model generates herding among agents' choices with the same observational features of similar phenomena produced by social learning, information-driven models. It stands as a competing explanation of how the behaviour of investors can in some instances bring about a currency or financial crisis.

Whether a herd is the result of an information problem or not has deep consequences. The informational herd theory implies that no market crash would happen if information were publicly available. On the contrary, in a coordination model agents' payoffs are affected by the actions of other agents. As a result, market crashes can occur even if information is fully accessible

Herding relies on two main assumptions. First, the bounded rationality of agents: they play a repeated coordination game, yet they only employ one-stage game strategies. Second, there exist a small probability  $\varepsilon$  that agents make mistakes when choosing the strategy.

In the limit, when the probability of an erroneous reply  $\varepsilon$  goes to zero, the model provides an equilibrium selection mechanism in the general class of 2x2 symmetric games. We prove that the strategy of a large population of repeatedly randomly coupled players engaging in a one-shot game converges in time to a Nash equilibrium, and that the arbitrarily small probability of a mistake results in a unique history-independent equilibrium. The long run equilibrium corresponds to the risk dominant one in the stage game. These results were first obtained by Kandori, Mailath and Rob (1993). The contribution of this paper is the adoption of an explicit individual selection mechanism to build the process describing the dynamic evolution of exactly the same multi-stage game. We introduce a flexible setup where the individual choice is explicitly modeled, where the population

dynamics is derived from the aggregation of the individual behaviour, and where a variety of selection/learning mechanism can be easily introduced.

Both the herding results and the equilibrium selection results obtain as well in a setup where agents make no mistake in choosing their strategy, but the payoff is affected by random stochastic shocks. Simply assume that for a player currently employing strategy  $i$  the payoff from choosing action  $j$  is the sum of a deterministic component (known at the time she moves), proportional to the number of individuals whose last action was  $j$ , and of a random component  $n_j * \theta_{i,j}$ , where  $n_j$  is the number of agents who chose strategy  $j$  and  $\theta_{ij}$  is drawn from the same random distribution for all agents. In this case the player will not take the action that has been most popular if the stochastic component attached to the other action is large enough.

Bergin and Lipman (1996) showed that the long run predictions of the model depend on the way mutation rates may change across states. It is left for future work to characterize the results in this paper under different assumptions for the state-dependent mutation rates  $\varepsilon_i = \varepsilon_i(n_i)$ .

## References

- [1] Amir, M. and Berninghaus, S., (1996), "Another approach to mutation and learning in games", Games and Economic Behaviour, 14.
- [2] Babichenko, Y., (2013), "BEst-reply dynamics in large binary-choice anonymous games", Games and Economic Behaviour 81.
- [3] Banerjee, A., (1992), "A simple model of herd behaviour", Quarterly Journal of Economics, 107.
- [4] - and Weibull, J., (1992), "Evolution and rationality: some recent game-theoretic results", The Industrial Institute for Economic and Social Research Working Paper 345-1992.
- [5] Bergin, J. and Lipman, B., (1996) "Evolution with state dependent mutations", Econometrica, vol.64 n.4.
- [6] Binmore, K. and Samuelson, L., (1997), "Muddling through: noisy equilibrium selection", Journal of Economic Theory, 74.

- [7] -, Samuelson, L., and Vaughan, R., (1995), "Musical chairs: modeling noisy evolution", *Games and Economic Behaviour*, 11.
- [8] Burdzy, K., Frankel, D. and Pauzner, A., (1998), "Fast equilibrium selection by rational players living in a changing world", mimeo, Tel Aviv University.
- [9] Calvo, G. and Mendoza, E., (1996), "Mexico's balance of payment crisis: a chronicle of a death foretold", *Journal of International Economics* 41: 235-64.
- [10] Chang, R. and Velasco, A., (1999), 'Banks, debt maturity and financial crises', mimeo, New York University.
- [11] Chari, V. and Kehoe, P., (1999), "Financial crises as herds", mimeo, Federal Reserve Bank of Minneapolis.
- [12] Cox, D. and Miller, H., (1965), *The Theory of Stochastic Processes*, Chapman and Hall.
- [13] Eyster, E. and Rabin, M., (2010), "Naive herding in rich information settings", *American Economic Journal: Microeconomics* 2.
- [14] Foster, D. and Young, P., (1990), "Stochastic evolutionary game dynamics", *Theoretical Population Biology*, 38.
- [15] Gale, Douglas, (1996), 'Whate have we learned from social learning?', *European Economic Review* 40: 627-28.
- [16] Gardiner, C., (1985), *Handbook of stochastic methods*, Springer Verlag.
- [17] Gillespie, Daniel, (1992), *Markov Processes*, Academic Press.
- [18] Kandori, M., Mailath, G. and Rob, R., (1993), "Learning, Mutation and long run equilibria in games", *Econometrica* 1 vol.61.
- [19] Kreindler, G. and Young, H. P., (2013), "Fast convergence in evolutionary equilibrium selection", *Games and Economic Behaviour* 80.
- [20] Mailath, G., (1999), "Do people play Nash equilibrium?", *Journal of Economic Literature*, Vol. 36.

- [21] Morris, S. and Shin, H., (1995), "Informational events that trigger currency attacks", Federal Reserve Bank of Philadelphia Working Papers 95-24.
- [22] -, (1999), 'Risk management with interdependent choice', Oxford Review of Economic Policy, Vol 15 n. 3.
- [23] Obstfeld, M., (1986) "Rational and self-fulfilling Balance of payments crises", American Economic Review Vol. 76.
- [24] Rodrik , D. and Velasco, A., (1999), 'Short-term capital flows', mimeo, New York University.
- [25] Sachs, J., Tornell, A. and Velasco, A., (1996), "The Mexican peso crisis: sudden death or death foretold?", Journal of International Economics 41.
- [26] Samuelson, L., (1994), "Stochastic stability in games with alternative best replies", Journal of Economic Theory, 64.
- [27] Sawa, R., (2012), "Mutation rates and equilibrium selection under stochastic evolutionary dynamics", International Journal of Game Theory 41.
- [28] Spyrou, S., (2014), "Herding in Financial Markets", Review of Behavioral Finance 5:2.
- [29] Weidlich, Wolfgang, (1991), "Physics and the social sciences", Physics Reports 1 vol. 204.

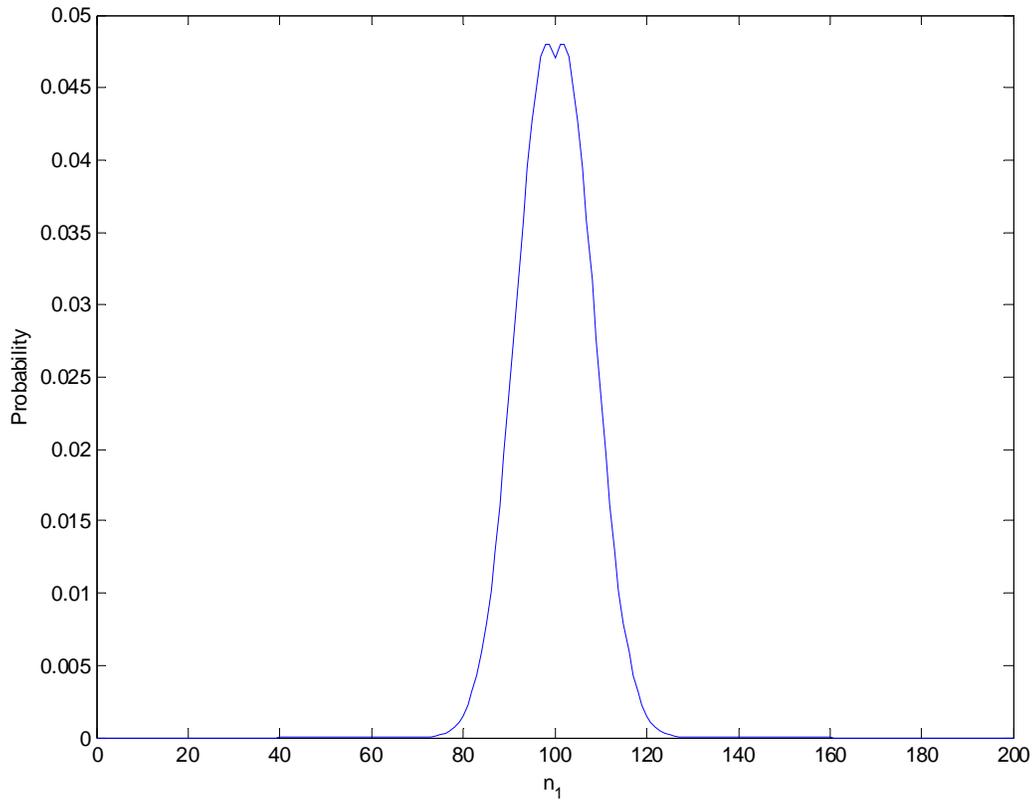


Figure 1: Coordination game with no Pareto-dominant strategy. Stable distribution of strategies for mutation rate  $\varepsilon = 0.95$ , payoffs  $a = 1$ ,  $d = 1$ , population size  $N = 200$ . Number  $n_1$  of agents choosing strategy  $a$ .

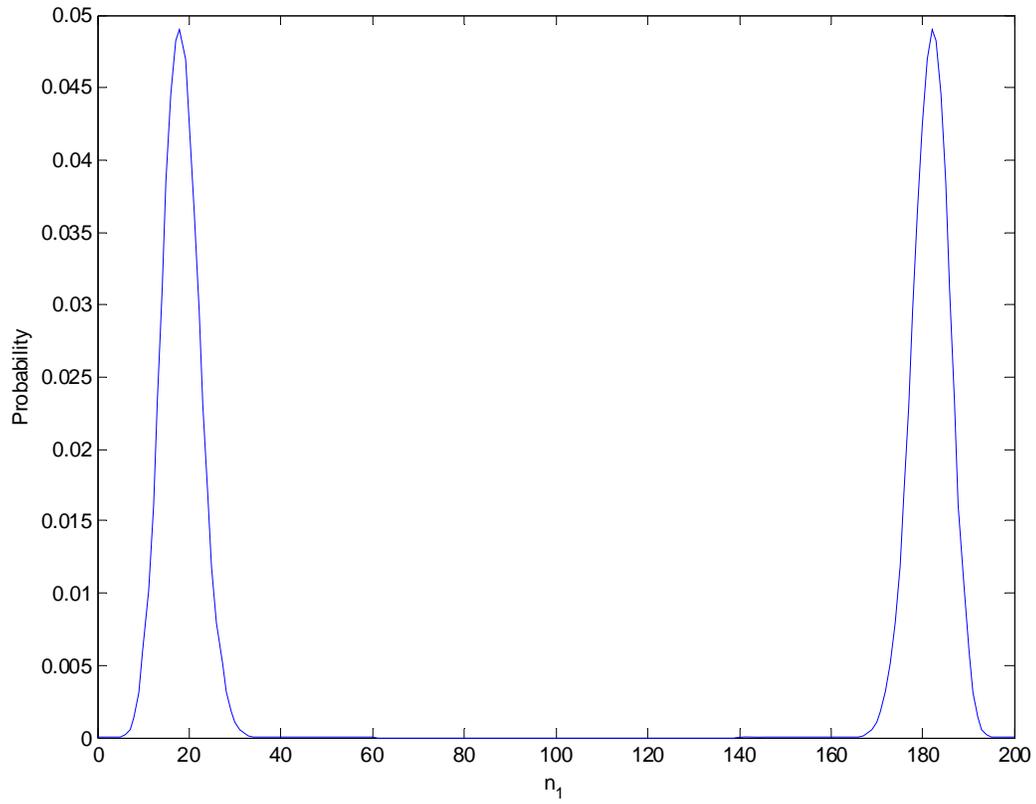


Figure 2: Coordination game with no Pareto-dominant strategy. Stable distribution of strategies for mutation rate  $\varepsilon = 0.1$ , payoffs  $a = 1$ ,  $d = 1$ , population size  $N = 200$ . Number  $n_1$  of agents choosing strategy  $a$ .

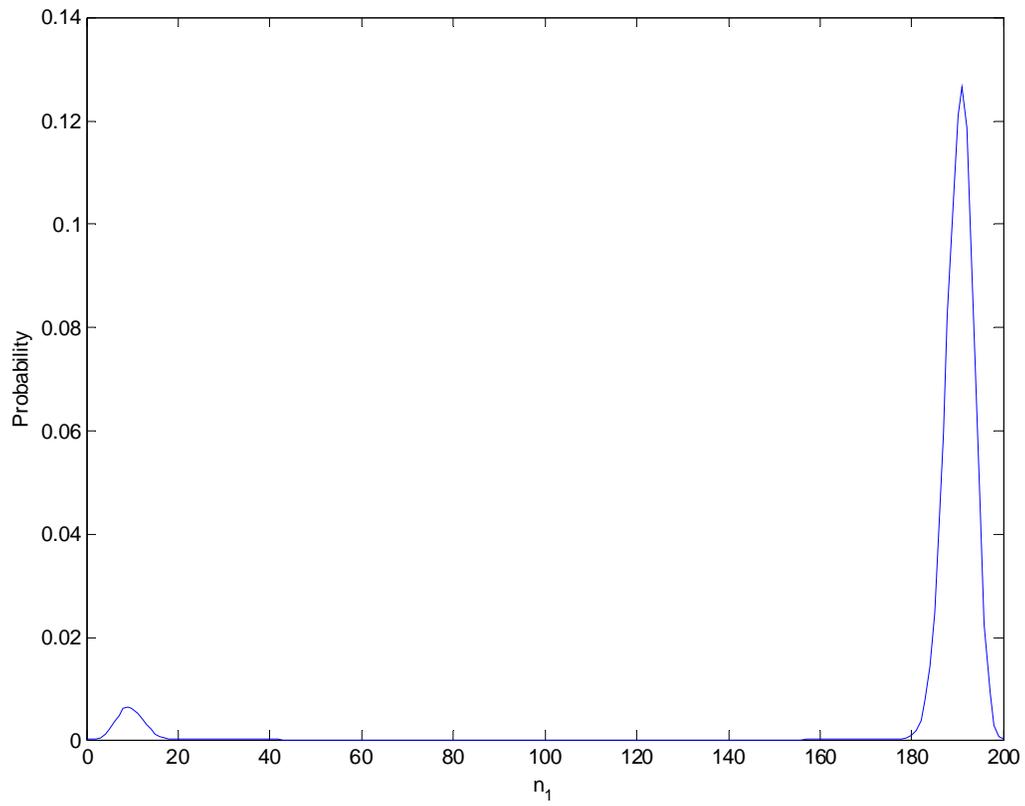


Figure 3: Coordination game with Pareto-dominant strategy. Stable distribution of strategies for mutation rate  $\varepsilon = 0.05$ , payoffs  $a = 1$ ,  $d = 0.99$ , population size  $N = 200$ . Number  $n_1$  of agents choosing strategy  $a$ .

# Appendix

## Proofs of Theorems

**Proof of Theorem 1** The birth-death process described by equation 5 fulfills the conditions of the birth-death stability theorem:

$$\left. \begin{array}{l} w_{12}(n-1) \\ \text{are both} \\ w_{21}(n) \end{array} \right| \begin{array}{l} > 0 \text{ if } n \in [1, N] \\ \\ = 0 \text{ otherwise} \end{array} \quad (15)$$

and:

$$\sum_{n=1}^N \prod_{j=1}^n \frac{w_{12}(j-1)}{w_{21}(j)} < \infty \quad (16)$$

Condition (16) is certainly met since  $N$  is assumed to be finite. Condition (15) simply states that the process is confined to a contiguous range of integer states and that the process be able to reach any state in  $[0, +N]^6$ . QED

**Proof of Theorem 4** Assume without loss of generality that  $s_1$  is the dominant strategy. Letting the state  $n = n_1$ , equation 11 implies:

$$\frac{w_+(n-1)}{w_-(n)} = \frac{(N-n+1)k(1-\epsilon)}{nk\epsilon} \quad \forall n$$

---

<sup>6</sup>Condition 15 in the case of a one-dimension state space ensures that the so-called detailed balance condition is met. That means that in the stable state not only

$$\begin{aligned} & [w_{21}(n+1) \Pr_s(n+1; t) + w_{12}(n-1) \Pr_s(n-1; t)] - \\ & [w_{12}(n) \Pr_s(n; t) + w_{21}(n) \Pr_s(n; t)] = 0 \end{aligned}$$

but the following also holds:

$$\begin{aligned} w_{21}(n+1) \Pr_s(n+1; t) - w_{12}(n) \Pr_s(n; t) &= 0 \\ w_{12}(n-1) \Pr_s(n-1; t) - w_{21}(n) \Pr_s(n; t) &= 0 \end{aligned}$$

In the case of multidimensional state space, a detailed balance condition is sufficient but not necessary for the existence of a stationary markov state density function. If detailed balance is not fulfilled, there is no general analytical method to build the stationary probability distribution, yet it can be numerically evaluated when it exist (Weidlich, 1991).

Equations 9 and 10 then imply:

$$\begin{aligned}
P_s(n) &= K \prod_{j=1}^n \left\{ \frac{(N-j+1)(1-\epsilon)}{j \epsilon} \right\} \\
&= K \left[ \frac{1-\epsilon}{\epsilon} \right]^n \frac{1}{n!} [N(N-1)(N-2)(N-3)\dots(N-n+1)] \\
&= K \left[ \frac{1-\epsilon}{\epsilon} \right]^n \frac{1}{n!} \binom{N}{n}
\end{aligned}$$

Define:

$$\theta_n = \frac{1}{n!} \binom{N}{n}$$

Therefore the stable distribution of the process is given by:

$$\begin{aligned}
P_s(n) &= K \left[ \frac{1-\epsilon}{\epsilon} \right]^n \theta_n \\
K &= \left[ 1 + \theta_1 \left( \frac{1-\epsilon}{\epsilon} \right) + \theta_2 \left( \frac{1-\epsilon}{\epsilon} \right)^2 + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^N \right]^{-1}
\end{aligned}$$

The limiting distribution for  $n = N$  is computed as  $\lim_{\epsilon \rightarrow 0} P_s(N)$  :

$$\lim_{\epsilon \rightarrow 0} \frac{\left[ \frac{1-\epsilon}{\epsilon} \right]^N \theta_N}{\left[ 1 + \theta_1 \left( \frac{1-\epsilon}{\epsilon} \right) + \theta_2 \left( \frac{1-\epsilon}{\epsilon} \right)^2 + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^N \right]} \quad (17)$$

Multiplying both terms of the ratio by  $\left( \frac{\epsilon}{1-\epsilon} \right)^N$  the above is equal to

$$\lim_{\epsilon \rightarrow 0} \frac{\theta_N}{\left[ \left( \frac{\epsilon}{1-\epsilon} \right)^N + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right)^{N-1} + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^{N-2} + \dots + \theta_N \right]} \quad (18)$$

As the mutation rate decreases to 0, all the terms in the denominator converge to 0 except the last one. Therefore:

$$P_s^*(N) = \lim_{\epsilon \rightarrow 0} P_s(N) = 1$$

The limiting distribution for  $n = 0$  is given by:

$$P_s^*(0) = \lim_{\epsilon \rightarrow 0} \frac{1}{\left[1 + \theta_1 \left(\frac{1-\epsilon}{\epsilon}\right) + \theta_2 \left(\frac{1-\epsilon}{\epsilon}\right)^2 + \dots + \theta_N \left(\frac{1-\epsilon}{\epsilon}\right)^N\right]} = 0$$

For any  $n \in (0, N)$ ,  $\lim_{\epsilon \rightarrow 0} P_s(n)$  is equal to:

$$\lim_{\epsilon \rightarrow 0} \frac{\left[\frac{1-\epsilon}{\epsilon}\right]^n \theta_n}{\left[1 + \theta_1 \left(\frac{1-\epsilon}{\epsilon}\right) + \theta_2 \left(\frac{1-\epsilon}{\epsilon}\right)^2 + \dots + \theta_N \left(\frac{1-\epsilon}{\epsilon}\right)^N\right]} \text{ or}$$

$$\lim_{\epsilon \rightarrow 0} \frac{\theta_n}{\left[\left(\frac{\epsilon}{1-\epsilon}\right)^n + \dots + \theta_n + \theta_{n+1} \left(\frac{1-\epsilon}{\epsilon}\right) + \dots + \theta_N \left(\frac{1-\epsilon}{\epsilon}\right)^{N-n}\right]}$$

The terms involving powers of  $\left(\frac{1-\epsilon}{\epsilon}\right)$  diverge to infinity, therefore:

$$P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n) = 0 \quad \forall n < N$$

The uniqueness of the limiting distribution follows from the uniqueness of  $P_s(n)$  for  $\epsilon > 0$ .

By the same line of reasoning the theorem statement can be proven when  $s_2$  is the dominant strategy. QED

**Proof of Theorem 5** *The stable distribution.* Let  $i^*$  be the largest integer in  $[0, N]$  such that  $i^* \leq z^*$ . Then equation 14 states that:

$$U_1(i) \leq U_2(i) \quad \forall i \leq i^*$$

$$U_1(i) > U_2(i) \quad \forall i > i^*$$

For  $n \leq i^*$  equation 11 implies:

$$\frac{w_+(n-1)}{w_-(n)} = \frac{(N-n+1)k\epsilon}{nk(1-\epsilon)}$$

Equations 9 and 10 then give:

$$P_s(n) = K \left[\frac{\epsilon}{1-\epsilon}\right]^n \theta_n \text{ for } 0 < n \leq i^*$$

For  $n = i^* + 1$  equation 11 implies:

$$\frac{w_+(n-1)}{w_-(n)} = \frac{(N-n+1)k\epsilon}{nk\epsilon}$$

Equations 9 and 10 then give:

$$P_s(n) = K \left[ \frac{\epsilon}{1-\epsilon} \right]^{n-1} \theta_n \text{ for } n = i^* + 1$$

For  $n > i^* + 1$  equation 11 implies:

$$\frac{w_+(n-1)}{w_-(n)} = \frac{(N-n+1)k(1-\epsilon)}{nk\epsilon}$$

Equations 9 and 10 then give:

$$\begin{aligned} P_s(n) &= K \left[ \frac{\epsilon}{1-\epsilon} \right]^{i^*} \left[ \frac{1-\epsilon}{\epsilon} \right]^{n-i^*-1} \theta_n \\ &= K \left[ \frac{1-\epsilon}{\epsilon} \right]^{n-1-2i^*} \theta_n \text{ for } n > i^* + 1 \end{aligned}$$

Note that this equation holds also for  $n = i^* + 1$ . It is now straightforward to calculate  $K$ :

$$K = \left[ 1 + \sum_{j=1}^{i^*} \theta_j \left[ \frac{\epsilon}{1-\epsilon} \right]^j + \sum_{j=i^*+1}^N \theta_j \left[ \frac{1-\epsilon}{\epsilon} \right]^{j-1-2i^*} \right]^{-1}$$

*The limiting distribution.* The limiting distribution for  $n = 0$  is given by:

$$P_s^*(0) = \lim_{\epsilon \rightarrow 0} K = \lim_{\epsilon \rightarrow 0} \frac{1}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{i^*} \left( \frac{1-\epsilon}{\epsilon} \right)^{-i^*} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-2i^*} \right]}$$

All the  $\left( \frac{\epsilon}{1-\epsilon} \right)$  power terms converge to 0. If  $s_1$  is the risk dominant strategy,  $i^* < \frac{N-1}{2}$ , therefore at least the last  $\left( \frac{1-\epsilon}{\epsilon} \right)$  power term will diverge to infinity. Conversely, the exponent of all the  $\left( \frac{1-\epsilon}{\epsilon} \right)$  will be negative, and they will all converge to zero. Therefore:

$$\begin{aligned} P_s^*(0) &= \lim_{\epsilon \rightarrow 0} P_s(0) = 0 \text{ if } s_1 \text{ is risk dominant} \\ P_s^*(0) &= \lim_{\epsilon \rightarrow 0} P_s(0) = 1 \text{ if } s_2 \text{ is risk dominant} \end{aligned}$$

The limiting distribution for  $n \in (0, i^*]$  is computed as  $\lim_{\epsilon \rightarrow 0} P_s(n)$  :

$$\lim_{\epsilon \rightarrow 0} \frac{\theta_n \left( \frac{\epsilon}{1-\epsilon} \right)^n}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{i^*} \left( \frac{\epsilon}{1-\epsilon} \right)^{i^*} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-2i^*} \right]}$$

It has been previously seen that the denominator converges to either one or infinity. Since the numerator converges to zero, it holds that:

$$P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n) = 0 \text{ for } \forall 0 < n \leq i^*$$

The limiting distribution for  $n \geq i^* + 1$  is given by:

$$\lim_{\epsilon \rightarrow 0} \frac{\left[ \frac{\epsilon}{1-\epsilon} \right]^{i^*} \left[ \frac{1-\epsilon}{\epsilon} \right]^{n-i^*-1} \theta_n}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{i^*} \left( \frac{1-\epsilon}{\epsilon} \right)^{-i^*} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-2i^*} \right]}$$

Multiplying both terms of the ratio by  $\left( \frac{\epsilon}{1-\epsilon} \right)^{N-1-2i^*}$  the above is equal to:

$$\lim_{\epsilon \rightarrow 0} \left[ \frac{\epsilon}{1-\epsilon} \right]^{N-n} \theta_n \left[ \begin{array}{l} \left( \frac{\epsilon}{1-\epsilon} \right)^{N-1-2i^*} + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right)^{1+(N-1-2i^*)} + \dots + \theta_{i^*} \left( \frac{\epsilon}{1-\epsilon} \right)^{N-1-i^*} + \dots \\ + \theta_{i^*+1} \left( \frac{\epsilon}{1-\epsilon} \right)^{N-1-i^*} + \theta_{i^*+2} \left( \frac{\epsilon}{1-\epsilon} \right)^{N-2-i^*} + \dots + \theta_{N-1} \left( \frac{1-\epsilon}{\epsilon} \right) + \theta_N \end{array} \right]^{-1}$$

Assume  $s_1$  is the risk dominant strategy. Then  $i^* < \frac{n-1}{2}$ . By examining the exponents of the power terms it is clear that as the mutation rate decreases to 0, all the terms in the denominator converge to 0 except the last term  $\theta_N$ . Only for  $n = N$  the numerator does not converge to 0 but rather to  $\theta_n$ . On the contrary, if  $s_2$  is the risk dominant strategy,  $i^* > \frac{n-1}{2}$  and all the terms in the denominator will converge to zero, except the last term  $\theta_N$  and the first term which will always diverge to infinity. The numerator will still converge either to 0 or  $\theta_N$ . Therefore we can state:

$$P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n) = 0 \text{ for } \forall i^* + 1 \leq n < N$$

$$\begin{aligned}
P_s^*(N) &= \lim_{\epsilon \rightarrow 0} P_s(N) = 1 \text{ if } s_1 \text{ is risk dominant} \\
P_s^*(N) &= \lim_{\epsilon \rightarrow 0} P_s(N) = 0 \text{ if } s_2 \text{ is risk dominant}
\end{aligned}$$

The uniqueness of the limiting distribution follows from the uniqueness of  $P_s(n)$  for  $\epsilon > 0$ . QED

**Proof of Theorem 7** *The stable distribution.* The proof follows the same steps as the previous one. Since  $z^*$  is integer,  $i^* = z^* = \frac{N-1}{2}$  is the largest integer in  $[0, N]$  such that  $i^* \leq z^*$ . Then equation 14 states that:

$$\begin{aligned}
U_1(i) &\leq U_2(i) \quad \forall i \leq \frac{N-1}{2} \\
U_1(i) &> U_2(i) \quad \forall i > \frac{N-1}{2}
\end{aligned}$$

For  $n \leq \frac{N-1}{2}$  equations 11, 9 and 10 give:

$$P_s(n) = K \left[ \frac{\epsilon}{1-\epsilon} \right]^n \theta_n \text{ for } 0 < n \leq \frac{N-1}{2}$$

For  $n \geq \frac{N+1}{2}$  equations 11, 9 and 10 give:

$$P_s(n) = K \left[ \frac{1-\epsilon}{\epsilon} \right]^{n-1-2i^*} \theta_n \text{ for } n \geq \frac{N+1}{2}$$

It is now straightforward to calculate  $K$ :

$$K = \left[ 1 + \sum_{j=1}^{\frac{N-1}{2}} \theta_j \left[ \frac{\epsilon}{1-\epsilon} \right]^j + \sum_{j=\frac{N+1}{2}}^N \theta_j \left[ \frac{1-\epsilon}{\epsilon} \right]^{j-1-(N-1)} \right]^{-1}$$

*The limiting distribution.* The limiting distribution for  $n = 0$  is given by:

$$P_s^*(0) = \lim_{\epsilon \rightarrow 0} K = \lim_{\epsilon \rightarrow 0} \frac{1}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{\frac{N-1}{2}} \left( \frac{\epsilon}{1-\epsilon} \right)^{\frac{N-1}{2}} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-(N-1)} \right]}$$

All the  $\left( \frac{\epsilon}{1-\epsilon} \right)$  power terms converge to 0, and so will all the  $\left( \frac{1-\epsilon}{\epsilon} \right)$  terms, except the

last one, converging to  $\theta_N = 1$ . Therefore the denominator converges to 2 and:

$$P_s^*(0) = \lim_{\epsilon \rightarrow 0} P_s(0) = 1/2 \text{ if } z^* = \frac{N-1}{2}$$

The limiting distribution for  $n \in (0, \frac{N-1}{2}]$  is computed as  $\lim_{\epsilon \rightarrow 0} P_s(n)$  :

$$\lim_{\epsilon \rightarrow 0} \frac{\theta_n \left( \frac{\epsilon}{1-\epsilon} \right)^n}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{\frac{N-1}{2}} \left( \frac{1-\epsilon}{\epsilon} \right)^{-\frac{N-1}{2}} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-(N-1)} \right]}$$

I established already that the denominator converges to 2. Since the numerator converges to zero, it holds that:

$$P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n) = 0 \text{ for } \forall 0 < n \leq \frac{N-1}{2}$$

The limiting distribution for  $n \geq \frac{N+1}{2}$  is given by:

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{\left[ \frac{\epsilon}{1-\epsilon} \right]^{\frac{N-1}{2}} \left[ \frac{1-\epsilon}{\epsilon} \right]^{n-\frac{N+1}{2}} \theta_n}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{\frac{N-1}{2}} \left( \frac{1-\epsilon}{\epsilon} \right)^{-\frac{N-1}{2}} + \dots + \theta_N \left( \frac{1-\epsilon}{\epsilon} \right)^{N-1-(N-1)} \right]} \\ = & \lim_{\epsilon \rightarrow 0} \frac{\left[ \frac{\epsilon}{1-\epsilon} \right]^{N-n} \theta_n}{\left[ 1 + \theta_1 \left( \frac{\epsilon}{1-\epsilon} \right) + \theta_2 \left( \frac{\epsilon}{1-\epsilon} \right)^2 + \dots + \theta_{\frac{N-1}{2}} \left( \frac{1-\epsilon}{\epsilon} \right)^{-\frac{N-1}{2}} + \dots + \theta_N \right]} \end{aligned}$$

The denominator of the expression in the limit equals 2. Only for  $n = N$  the numerator does not converge to 0 but rather to  $\theta_N = 1$ . Therefore we can state:

$$P_s^*(n) = \lim_{\epsilon \rightarrow 0} P_s(n) = 0 \text{ for } \forall \frac{N+1}{2} \leq n < N$$

$$P_s^*(N) = \lim_{\epsilon \rightarrow 0} P_s(N) = 1/2 \text{ if } z^* = \frac{N-1}{2}$$

The uniqueness of the limiting distribution follows from the uniqueness of  $P_s(n)$  for  $\epsilon > 0$ . QED